

# FLAG KERNELS OF ARBITRARY ORDER

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**ABSTRACT.** We extend the notion of a flag kernel to distributions of arbitrary order. Under some natural restrictions on order, we get a kind of functional calculus for thus generalized flag kernels.

## INTRODUCTION

In the study of flag kernels it is sometimes convenient to work with a broader class of kernels, cf. [4] and [5]. In this paper we introduce a class of tempered distributions generalizing homogenous distributions smooth away from the origin and containing the flag kernels of Nagel-Ricci-Stein [6] as objects of order 0. As a preliminary step it seems only natural to consider first what we might call Calderón-Zygmund kernels of arbitrary order.

Let  $a \in \mathbf{R}$ . We say that a distribution  $P \in \mathcal{S}^*(V)$  is a  $\mathcal{F}(a)$ -kernel if it is smooth away from the origin and satisfies the following conditions:

*Size condition:* For every multiindex  $\alpha$ ,

$$(0.1) \quad |D^\alpha P(x)| \leq C_\alpha |x|^{a-|\alpha|}, \quad x \neq 0.$$

*Cancellation condition:* There exists a norm  $\|\cdot\|$  in the Schwartz space  $\mathcal{S}(V)$  such that for every  $\varphi \in \mathcal{S}(V)$  and every  $R > 0$

$$(0.2) \quad \left| \int \varphi(Rx) P(x) dx \right| \leq R^a \|\varphi\|.$$

In a similar fashion (see below) we generalize the notion of a flag kernel.

Our goal is to extend the usual convolution of functions on a homogenous group to the class of thus generalized flag kernels. This is not always possible as can be seen from simple examples. However, under some natural restrictions on order, we get a kind of functional calculus. The classes of flag kernels considered here have their smooth counterparts in the classes described in [3] whose properties may be transferred into the new context under study. Let us add that this nonsmooth calculus has already proved useful in the proof of the  $L^p$ -boundedness of flag kernels in [5]. The reader may wish to compare our results with those of Geller-Coré [1] where a very different approach has been taken.

The techniques used in the first section are a slight extension of those of Nagel-Ricci-Stein [6]. All proofs are included for the sake of completeness. On the background theory of homogeneous groups the reader is referred to Folland-Stein [2].

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## 1. FLAG KERNELS

Let  $V$  be a finite-dimensional real vector space with a fixed family of non-isotropic dilations  $x \rightarrow tx$ ,  $t > 0$ , their exponents of homogeneity being

$$1 = p_1 < p_2 < \cdots < p_d.$$

We have

$$V = \bigoplus_{j=1}^d V_j, \quad V_j = \{x \in V : tx = t^{p_j}x\}.$$

The homogenous norm  $|\cdot|$  on  $V$  induces “partial” homogeneous norms

$$|x|_j = |(x_1, x_2, \dots, x_j, 0, \dots, 0)|, \quad 1 \leq j \leq d,$$

corresponding to the filtration

$$(1.1) \quad V_{(k)} = \bigoplus_{j=1}^k V_j, \quad 1 \leq k \leq d.$$

In particular,  $|x|_1 = |x_1|$ , and  $|x|_d = |x|$ . The homogeneous dimension of  $V$  is

$$Q = \sum_{j=1}^d Q_j, \quad Q_j = p_j \dim V_j.$$

There exists a corresponding homogeneous structure on the dual space  $V^*$  so a similar notation applies to  $V^*$ . However, here we let

$$|\xi|_j = |(0, \dots, 0, \xi_j, \xi_{j+1}, \dots, \xi_d)|, \quad 1 \leq j \leq d,$$

the corresponding filtration being

$$(1.2) \quad V_{(k)}^* = \bigoplus_{j=k}^d V_j^*, \quad 1 \leq k \leq d.$$

In particular,  $|\xi|_1 = |\xi|$ , and  $|\xi|_d = |\xi|_d$ .

In expressions like  $D^\alpha$  or  $x^\alpha$  we shall use multiindices

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d),$$

where

$$\alpha_k = (\alpha_{k1}, \alpha_{k2}, \dots, \alpha_{kn_k}), \quad n_k = \dim V_k = \dim V_k^*,$$

are themselves multiindices with positive integer entries corresponding to the spaces  $V_k$  or  $V_k^*$ . The homogeneous length of  $\alpha$  is defined by

$$|\alpha| = \sum_{k=1}^d |\alpha_k|, \quad |\alpha_k| = n_k p_k.$$

We define the class  $\mathcal{F}(\nu)$  by induction on the homogeneous step  $d$ . If  $d = 1$ , that has been already done in Introduction. If  $d \geq 2$  and  $\nu = (\nu_1, \dots, \nu_d)$ , then  $P \in \mathcal{F}(\nu)$  if  $P$  is a tempered distribution smooth away from the hyperspace  $x_1 = 0$  and satisfying

$$(1.3) \quad |D^\alpha P(x)| \leq C_\alpha |x|_1^{-\nu_1 - Q_1 - |\alpha_1|} |x|_2^{-\nu_2 - Q_2 - |\alpha_2|} \dots |x|_d^{-\nu_d - Q_d - |\alpha_d|}$$

for  $x_1 \neq 0$ . Moreover, for any  $1 \leq j \leq d$ , there exists a norm  $\|\cdot\|$  in  $\mathcal{S}(V_j)$  such that

$$(1.4) \quad \langle P_{R,\varphi}, f \rangle = R^{-\nu_j} \int_V \varphi(Rx_j) f(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d) P(x) dx$$

is in  $\mathcal{F}(\nu')$  on  $V_1 \oplus \dots \oplus V_{j-1} \oplus V_{j+1} \oplus \dots \oplus V_d$ , where

$$\nu' = (\nu_1, \dots, \nu_{j-1}, \nu_{j+1}, \dots, \nu_d),$$

and this is uniform in  $\varphi \in \mathcal{S}(V_j)$  of norm 1 and  $R > 0$ . Let us remark that if  $\nu_j < 0$  for all  $j$ , then (1.3) implies (1.4). The definition is relative to the filtration (1.1). The flag kernels on  $V^*$  will be understood as relative to (1.2).

It follows directly from the definition that

**1.5. Lemma.** *If  $P \in \mathcal{F}(\nu)$ , then  $x^\alpha D^\beta P \in \mathcal{F}(\mu)$ , where  $\mu_k = \nu_k + |\alpha_k| - |\beta_k|$ .*

The next lemma is an easy corollary to the Arzela-Ascoli theorem.

**1.6. Lemma.** *Let  $\varphi \in C_c^\infty(V)$  be equal to 1 in a neighbourhood of 0. For  $\varepsilon > 0$ , let  $\varphi_\varepsilon(x) = \varphi(\varepsilon x)$ . Let  $P \in \mathcal{F}(\nu)$ . Then  $P_\varepsilon = \varphi_\varepsilon P$  are a uniformly bounded family of compactly supported distributions in  $\mathcal{F}(\nu)$ . Moreover,  $P_\varepsilon \rightarrow P$  in the sense of distributions, and  $\widehat{P}_\varepsilon(\xi) \rightarrow \widehat{P}(\xi)$  in  $C^\infty(V^*)$  away from the hyperspace  $\xi_d = 0$  as  $\varepsilon \rightarrow 0$ .*

The following proposition is the main objective of this section.

**1.7. Proposition.**  *$P \in \mathcal{F}(\nu)$  implies  $\widehat{P} \in \mathcal{F}(\mu)$ , where  $\mu_k = -Q_k - \nu_k$ .*

*Proof.* In view of Lemma 1.6, we may assume that  $P$  has compact support. Then  $\widehat{P}$  is a smooth function, and  $D^\alpha \widehat{P}$  is a Fourier transform of  $(-ix)^\alpha P \in \mathcal{F}(\nu_1 - |\alpha_1|, \dots, \nu_d - |\alpha_d|)$  so it is enough to show that

$$|\widehat{P}(\xi)| \leq C |\xi|_1^{\nu_1} \dots |\xi|_d^{\nu_d}, \quad |\xi_d| > 0,$$

and check the cancellation condition.

Let us first consider the case  $d = 1$ . Let  $\nu = a$ . Let  $\varphi$  be a compactly supported smooth function equal to 1 on the support of  $P$ . Then for every  $|\xi| \geq 1$

$$\widehat{P}(\xi) = \int e^{ix\xi} P(x) dx = \int e^{ix\xi} \varphi(|\xi|x) P(x) dx,$$

and

$$e^{ix\xi} \varphi(|\xi|x) = \psi_\xi(|\xi|x),$$

where the family  $\psi_\xi$  is bounded in  $\mathcal{S}(V)$ . Therefore, by (0.2),

$$(1.8) \quad |\widehat{P}(\xi)| \leq C |\xi|^a, \quad |\xi| \geq 1.$$

However, the dilates  $\langle P_R, f \rangle = \langle P, f \circ \delta_R \rangle$  satisfy the same estimate uniformly in  $R$  so (1.8) holds for  $|\xi| > 0$ . As for the cancellation property, by (0.1),

$$|\langle \widehat{P}, \varphi \circ \delta_R \rangle| = R^{-Q} |\langle P, \widehat{\varphi} \circ \delta_{R^{-1}} \rangle| \leq \|\varphi\| R^{-Q-a}$$

for a suitable norm in  $\mathcal{S}(V^*)$ .

Assume that our claim holds for  $1 \leq d' < d$  and let  $P \in \mathcal{F}(\nu)$  on a homogeneous group of step  $d$ . Let  $\xi \in V^*$  with  $|\xi_d| \geq 1$ . Let  $k$  be the smallest index such that  $|\xi_j| \leq |\xi_d|$  for  $k \leq j \leq d$ . We shall write

$$x = (x', x'') = (x_1, \dots, x_{k-1} | x_k, \dots, x_d),$$

and

$$\xi = (\xi', \xi'') = (\xi_1, \dots, \xi_{k-1} | \xi_k, \dots, \xi_d).$$

Therefore

$$(1.9) \quad |\xi'|_i = |\xi_i| + \dots + |\xi_{k-1}| \approx |\xi|_i, \quad |\xi''| = |\xi_k| + \dots + |\xi_d| \approx |\xi_d|$$

for  $1 \leq i \leq k-1$ .

Let  $\psi_j$  be a compactly supported smooth function in the  $x_j$ -variable, and let

$$\psi(x'') = \psi_k(x_k) \dots \psi_d(x_d).$$

We also assume that  $\psi P = P$ . The distribution

$$\begin{aligned} \langle P_1, f \rangle &= \int f(x') e^{-ix'' \xi''} P(x', x'') dx' dx'' \\ &= \int f(x') e^{-ix'' \xi''} \psi(|\xi''| x'') P(x', x'') dx' dx'' \end{aligned}$$

is in  $\mathcal{F}(\nu_1, \dots, \nu_{k-1})$  uniformly in  $R = |\xi''| \geq |\xi_d| \geq 1$  so that by induction hypothesis and (1.9)

$$\begin{aligned} |\widehat{P}(\xi)| &= |\widehat{P}_1(\xi')| \leq C |\xi''|^{\nu_k + \dots + \nu_d} |\xi'|_1^{\nu_1} \dots |\xi'|_{k-1}^{\nu_{k-1}} \\ &\leq C_1 |\xi|_1^{\nu_1} \dots |\xi|_{k-1}^{\nu_{k-1}} |\xi|_k^{\nu_k} \dots |\xi|_d^{\nu_d}, \quad |\xi_d| \geq 1. \end{aligned}$$

As before, we remove the restriction  $|\xi_d| \geq 1$  by invoking the uniformity of the dilates of  $P$ .

Let  $\varphi$  be a Schwartz class function in the  $\xi_j$ -variable. By induction hypothesis again, the distributions

$$R^{-\mu_j} \langle \widehat{P}_j, f \rangle = \int \varphi(R \xi_j) f(\xi_1, \dots, \xi_{j-1}, \xi_{j+1}, \dots, \xi_d) d\xi$$

which are the Fourier transforms of

$$R^{-\mu_j} \langle P_j, g \rangle = R^{\nu_j} \int \varphi^\vee(R^{-1} x_j) g(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d) dx,$$

are bounded in  $\mathcal{F}(\mu_1, \dots, \mu_{k-1}, \mu_k, \dots, \mu_d)$  uniformly in  $R > 0$  and  $\|\varphi^\vee\| = 1$  for a suitable norm in  $\mathcal{S}(V_j)$ , which proves the cancellation property and completes the proof.  $\square$

Let us observe that if  $A \in \mathcal{F}(\mu)$  and  $B \in \mathcal{F}(\nu)$  and  $\nu_j + \mu_j > -Q_j$ ,  $1 \leq j \leq d$ , then the product of their Fourier transforms, which is well defined almost everywhere, is locally integrable, and

$$\widehat{A} \cdot \widehat{B} \in \mathcal{F}(-Q_1 - \mu_1 - \nu_1, \dots, -Q_d - \mu_d - \nu_d),$$

and therefore, by Proposition 1.7,  $\widehat{A} \cdot \widehat{B} = \widehat{C}$ , where  $C \in \mathcal{F}(\mu + \nu)$ . By Lemma 1.6,

$$C = \lim_{\varepsilon \rightarrow 0} A_\varepsilon \star B,$$

so it can be regarded as an extended convolution of  $A$  and  $B$ . Of course,  $C = A \star B$ , if, e.g.  $A$  has compact support.

In the next section we show that it is still so if the vector group structure on  $V$  is replaced by that of a noncommutative nilpotent group as long as it is congruent with the homogeneous structure.

## 2. CONVOLUTION

From now on we assume that  $V$  is a homogeneous Lie algebra with respect to the dilations which are now supposed to be automorphisms of  $V$ . We have

$$[V_i, V_j] \subset \begin{cases} V_k, & \text{if } p_i + p_j = p_k, \\ \{0\}, & \text{if } p_i + p_j \notin \mathcal{P}, \end{cases}$$

where  $\mathcal{P} = \{p_j : 1 \leq j \leq d\}$ .

We shall also regard  $V$  as a Lie group with the Campbell-Hausdorff multiplication

$$xy = x + y + r(x, y),$$

where  $r(x, y)$  is the (finite) sum of terms of order at least 2 in the Campbell-Hausdorff series for  $V$ .

We say that a tempered distribution  $A$  on  $V$  belongs to the class  $S(\nu)$ , if its Fourier transform  $\hat{A}$  is smooth and satisfies the estimates

$$|D^\alpha \hat{A}(\xi)| \leq (1 + |\xi|_1)^{\nu_1 - |\alpha_1|} \dots (1 + |\xi|_d)^{\nu_d - |\alpha_d|}, \quad \text{all } \alpha.$$

The space  $S(\nu)$  is a Fréchet space if endowed with the family of seminorms

$$A \rightarrow \sup_{|\alpha| \leq N} \sup_{\xi \in V^*} (1 + |\xi|_1)^{-\nu_1 + |\alpha_1|} \dots (1 + |\xi|_d)^{-\nu_d + |\alpha_d|} |D^\alpha \hat{A}(\xi)|,$$

for  $N \in \mathbf{N}$ . Apart from the Fréchet convergence, it is convenient to consider the  $C^\infty$ -convergence on Fréchet bounded subsets of  $S(\nu)$ .

**2.1. Proposition.** *The mapping*

$$S(\mu) \times S(\nu) \ni (A_1, A_2) \rightarrow A_1 \star A_2 \in S(\mu + \nu)$$

*defined initially for the Schwartz functions extends uniquely to a mapping which is continuous when all the spaces are endowed with either of the above defined convergences.*

*Proof.* This follows from Corollary 5.2 of [3]. □

Let  $A$  be a distribution on  $V$ . Let  $\varphi \in C_c^\infty(V)$  be equal to 1 in a neighbourhood of 0. Recall that for  $\varepsilon > 0$ , we write  $\varphi_\varepsilon(x) = \varphi(\varepsilon x)$  and  $A_\varepsilon = \varphi_\varepsilon A$ . The following is the main result of this paper. Note that the restriction we impose in the second part of the statement refers to the sum of the exponents and not to the exponents themselves.

**2.2. Theorem.** *Let  $A \in \mathcal{F}(\nu)$  and  $B \in \mathcal{F}(\mu)$ . Then the functions*

$$\Phi_\varepsilon(\xi) = \widehat{A_\varepsilon \star B}(\xi)$$

are convergent in  $C^\infty(V^*)$  away from the hyperspace  $\xi_d = 0$  to a smooth function  $\Phi$  satisfying the estimates

$$|D^\alpha \Phi(\xi)| \leq C_\alpha |\xi|_1^{\mu_1 + \nu_1 - |\alpha_1|} \dots |\xi|_d^{\mu_d + \nu_d - |\alpha_d|},$$

where the bounds depend only on those of  $A$  in  $\mathcal{F}(\nu)$  and of  $B$  in  $\mathcal{F}(\mu)$ . If, moreover,  $\mu_k + \nu_k > -Q_k$  for all  $1 \leq k \leq d$ , then  $\Phi$  is locally integrable, and

$$\lim_{\varepsilon \rightarrow 0} A_\varepsilon \star B = \Phi^\vee \in \mathcal{F}(\mu + \nu),$$

where the limit is understood in the sense of distributions.

*Proof.* Let us first assume that  $A$  has compact support. Let  $h \in C_c^\infty(V_d^*)$  be supported where  $1/2 \leq |\xi_d| \leq 4$  and equal to 1 where  $1 \leq |\xi_d| \leq 2$ . Let

$$\widehat{A}_1(\xi) = h(\xi) \widehat{A}(\xi), \quad \widehat{B}_1(\xi) = h(\xi) \widehat{B}(\xi).$$

Then  $A_1 \in S(\nu)$ ,  $B_1 \in S(\mu)$ , and, by Proposition 2.1, there exist constants  $C_\alpha$  such that

$$\begin{aligned} |D^\alpha(\widehat{A \star B})(\xi)| &= |D^\alpha(\widehat{A_1 \star B_1})(\xi)| \\ &\leq C_\alpha |\xi|_1^{\nu_1 + \mu_1 - |\alpha_1|} \dots |\xi|_d^{\nu_d + \mu_d - |\alpha_d|}, \quad 1 \leq |\xi_d| \leq 2. \end{aligned}$$

Note that  $1 + |\xi| \approx |\xi|$  on the set where  $|\xi_d| \approx 1$ . The families  $A_t(x) = t^{-Q-n} A(t^{-1}x)$  and  $B_t(x) = t^{-Q-m} B(t^{-1}x)$ , where  $n = \sum_{k=1}^d \nu_k$ ,  $m = \sum_{k=1}^d \mu_k$ , satisfy the estimates

$$|D^\alpha \widehat{A_t}(\xi)| \leq C_\alpha |\xi|_1^{\nu_1 - |\alpha_1|} \dots |\xi|_d^{\nu_d - |\alpha_d|}$$

and

$$|D^\alpha \widehat{B_t}(\xi)| \leq C_\alpha |\xi|_1^{\mu_1 - |\alpha_1|} \dots |\xi|_d^{\mu_d - |\alpha_d|}$$

uniformly in  $t > 0$  so that

$$|D^\alpha \widehat{A \star B}(\xi)| \leq C_\alpha |\xi|_1^{\nu_1 + \mu_1 - |\alpha_1|} \dots |\xi|_d^{\nu_d + \mu_d - |\alpha_d|}, \quad t \leq |\xi_d| \leq 2t,$$

for arbitrary  $t > 0$ .

It follows that  $\widehat{A_\varepsilon \star B}$  satisfy the above estimates uniformly in  $\varepsilon > 0$  and converge to some  $\Phi$  in  $C^\infty$  on  $V^*$  away from  $\xi_d = 0$ . The other part of the claim follows by Proposition 1.7.  $\square$

**2.3. Corollary.** *Let  $A \in \mathcal{F}(\nu)$  and  $B \in \mathcal{F}(\mu)$ . If  $A$  is compactly supported, and  $\mu_k + \nu_k > -Q_k$  for all  $1 \leq k \leq d$ , then  $A \star B \in \mathcal{F}(\mu + \nu)$ .*

Let us consider the following example, where  $V = \mathbf{R}$ . Let

$$\langle \widehat{A}, f \rangle = \int_{|\xi| \leq 1} \frac{f(\xi) d\xi}{\sqrt{|\xi|}}.$$

Then  $A \in \mathcal{F}(-1/2)$ , but “ $A \star A$ ” does not satisfy the cancellation condition. In fact, let  $f$  be supported in  $[-1, 1]$ . The function  $\Phi$  from the proof of Theorem 2.2 is here equal to  $\Phi(\xi) = |\xi|^{-1}$ . Any extension of  $\Phi$  to a distribution on  $\mathbf{R}$  is of the form

$$\langle \widehat{B}, f \rangle = cf(0) + \int_{|\xi| \leq 1} \frac{f(\xi) - f(0)}{\xi} d\xi + \int_{|\xi| \geq 1} \frac{f(\xi) d\xi}{\xi}$$

so that, for  $R > 1$ ,

$$\langle \widehat{B}, f \circ \delta_R \rangle = cf(0) + \int_{|\xi| \leq 1} \frac{(f(\xi) - f(0)) d\xi}{|\xi|} - 2 \log R \cdot f(0).$$

The last expression is unbounded as  $R \rightarrow \infty$ , which shows that  $B$  does not satisfy the cancellation condition.

We can extend the law of composition to the distributions  $A$  on  $V$  such that  $\widehat{A}$  is smooth away from the hyperspace  $\xi_d = 0$  and

$$|D^\alpha A(\xi)| \leq C_\alpha |\xi|_1^{\nu_1 - |\alpha_1|} \cdots |\xi|_d^{\nu_d - |\alpha_d|}, \quad \xi_d \neq 0,$$

for some  $\nu = (\nu_1, \dots, \nu_d)$ . Let us denote the class of such distributions by  $\mathcal{F}_0(\nu)$ . Elements of  $\mathcal{F}_0(\nu)$  can be interpreted as operators on a suitably chosen space of test functions. In fact, let  $\mathcal{S}_d(V)$  denote the closed subspace of  $\mathcal{S}(V)$  consisting of functions  $f$  with

$$\int_{V_d} f(x) p(x_d) dx_d = 0$$

for all polynomials in the  $x_d$ -variable. It is not hard to see that every  $A \in \mathcal{F}_0(\nu)$  can be regarded as a continuous linear form on this space acting by

$$\langle A, f \rangle = \lim_{\varepsilon \rightarrow 0} \int_{|x_1| \geq \varepsilon} f(x) A(x) dx.$$

Moreover, the convolution operator

$$T_A(x) = f \star \widetilde{A}(x) = \int f(xy) A(dy)$$

is continuous. If  $A, B$  belong to  $\mathcal{F}_0(\nu)$ , then  $T_A = T_B$  if and only if  $\widehat{A} - \widehat{B}$  is supported where  $\xi_d = 0$ . Therefore, strictly speaking, the convolution operators  $T_A$  correspond to the classes of equivalent distributions.

**2.4. Corollary.** *Let  $A \in \mathcal{F}_0(\nu)$  and  $B \in \mathcal{F}_0(\mu)$ . Then there exists a  $C \in \mathcal{F}_0(\nu + \mu)$  such that*

$$T_A \circ T_B = T_C.$$

*Proof.* It is sufficient to observe that the argument of the proof of Theorem 2.2 works here as well and yields a function  $\Phi$  satisfying the desired estimates. Moreover,  $\Phi$  admits extension to a distribution on  $V^*$ .  $\square$

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